Quadratic test study guide

Use this as a reminder of the things you should know for the test. It is not a substitute for practice!

The test covers sections 5.1-5.6 in the book.

The forms (e.g. standard form, vertex form, intercept form etc.) are very precise. For example, the vertex form is \( y = a(x - h)^2 + k \). If you have something like \( 2(x + 3)^2 - 7 \) that is not exactly in the form! You should force the operation after the \( x \) to be subtraction and the operation before \( k \) to be addition, converting \( y = 2(x + 3)^2 - 7 \) to \( y = 2(x - (-3))^2 + (-7) \). Now it’s exactly in vertex form so you can say that the vertex is \((-3, -7)\). You may do this in your head, but unless you are sure of yourself, write it down to make errors less likely. For another example, the quadratic expression \( x^2 - 2x + 3 \) can be rewritten as \( 1x^2 + (-2)x + 3 \) to exactly fit standard form (see below).

Quadratic expression: an expression that can be written in the form \( ax^2 + bx + c \).

Quadratic function: a function that can be written in the form \( y = ax^2 + bx + c \) (or \( f(x) = ax^2 + bx + c \)). This form is also called the standard form.

Quadratic equation: an equation that can be written in the form \( 0 = ax^2 + bx + c \).

Quadratic formula: The formula that gives the solution(s) of a quadratic equation in standard form:
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Note the similarities and differences between an quadratic expressions, equations, functions and the quadratic formula.

The graph of a quadratic function in standard form is a parabola:

- Opens up if \( a > 0 \), down if \( a < 0 \)
- Wider than \( y = x^2 \) if \( |a| < 1 \) and narrower if \( |a| > 1 \).
- \( x \)-coordinate of vertex is \(-\frac{b}{2a}\) (note similarity with quadratic formula)
- Axis of symmetry is the vertical line \( x = \frac{b}{2a} \)

Other forms of quadratic functions:

- Vertex form: \( y = a(x - h)^2 + k \). The vertex is \((h, k)\). Axis of symmetry is \( x = h \).
- Intercept form: \( y = a(x - p)(x - q) \). The \( x \)-intercepts (also called zeroes) are \( p \) and \( q \). Axis of symmetry is halfway between \( p \) and \( q \): \( x = \frac{p + q}{2} \)
- Both forms: opens up if \( a > 0 \), down if \( a < 0 \)

Factoring quadratic expressions:

- To factor \( x^2 + bx + c \), find (if you can) \( m \) and \( n \) where \( b = m + n \) and \( c = mn \). Then \((x + m)(x + n) = x^2 + bx + c\). Example: To factor \( x^2 + 5x + 6 \), note that \( 5 = 2 + 3 \) and \( 6 = 2 \cdot 3 \). Thus \( x^2 + 5x + 6 = (x + 2)(x + 3) \)
- To factor \( ax^2 + bx + c \) it’s harder; keep trying until you get something that multiplies out correctly. Example: \( 3x^2 - 17x + 10 = (3x - 2)(x - 5) \)
• Special patterns:
  ◦ **Difference of two squares**: \(a^2x^2 - b^2 = (a + b)(a - b)\). Example: \(x^2 - 9 = (x + 3)(x - 3)\)
  ◦ **Perfect square trinomial**: \(x^2 + 2px + p^2 = (x + p)^2\). Example: \(x^2 + 6x + 9 = (x + 3)^2\)

**Zero product property**: If \(ab = 0\) then \(a = 0\) or \(b = 0\) (or both). This enables us to solve quadratic equations by factoring.

**Solving an equation**: finding values of variable(s) that make it true.

We can use the zero product property to **solve a quadratic equation by factoring**:

Example: Solve \(x^2 + 3x - 18 = 0\). Solution: Factor as \((x + 6)(x - 3) = 0\). Zero product property tells us that \(x + 6 = 0\) or \(x - 3 = 0\). Thus the solutions (values of \(x\) that make it true) are \(-6\) and \(3\).

**Finding the zeros of a quadratic function** is just solving the equation that sets the function value to zero. For example, finding the zeros of the function \(y = x^2 + 3x - 18\) is just solving \(x^2 + 3x - 18 = 0\) (see above).

A number \(r\) is a **square root** of a number \(s\) if \(r^2 = s\). Every positive real number has two square roots. The positive square root of \(s\) is written \(\sqrt{s}\); the negative square root is written \(-\sqrt{s}\). The symbol \(\sqrt{\cdot}\) is called the **radical sign** and the number under it is the **radicand**. An expression with a radical sign and radicand underneath, like \(\sqrt{s}\), is a **radical**.

If \(a \geq 0\) and \(b \geq 0\), then we have the **Product property**: \(\sqrt{ab} = \sqrt{a} \sqrt{b}\) and **Quotient property**: \(\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\).

Note that there are no similar sum and difference properites: \(\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}\) and \(\sqrt{a - b} \neq \sqrt{a} - \sqrt{b}\). For example, \(\sqrt{1 + 4} \neq \sqrt{1} + \sqrt{4}\) (work it out).

**Solving quadratic equations using square roots**: Try to get a square on both sides of the equation and then take square roots, being careful about positive and negative. Usually one side is an obvious square and the other side is a number, which is also an obvious, if ugly, square. Example:

\[
\begin{align*}
2x^2 + 1 &= 17 \\
2x^2 &= 16 \\
x^2 &= 8 \\
x &= \pm \sqrt{8} \\
x &= \pm 2\sqrt{2}
\end{align*}
\]

**Complex numbers**

\(i\)

Define \(i = \sqrt{-1}\), so \(i^2 = -1\).

Properties:

• \(\sqrt{-1} = i \sqrt{1}\)

• For example, \(i \sqrt{3} = \sqrt{-3}\)
Example: solve quadratic equation

Solve \(3x^2 + 10 = -26\).

\[
\begin{align*}
3x^2 + 10 &= -26 \\
3x^2 &= -36 \\
x^2 &= -12 \\
x &= \pm \sqrt{-12} \\
x &= \pm i\sqrt{12} \\
x &= \pm i\sqrt{3} \\
x &= \pm 2i\sqrt{3}
\end{align*}
\]

Standard form of complex number

\(a + bi\)

Where \(a, b\) are real numbers. \(a\) is the real part and \(bi\) is the imaginary part.

Arithmetic

Adding/subtracting: Just think of \(i\) as a variable, like \(x\):

\[
(4 + 5i) + (3 + 2i) = (4 + 3 + 5i + 2i) = (7 + 7i)
\]

Multiplying: Think of \(i\) as a variable, except that it can do tricks.

\[
(4 + 5i)(3 + 2i) = 4 \cdot 3 + 4 \cdot 2i + 3 \cdot 5i + 5 \cdot 2i^2 = 12 + 8i + 15i + 10i^2 = 12 + 23i + (-10) = 2 + 23i
\]

Here is the same multiplication using the distributive law rectangle:

<table>
<thead>
<tr>
<th>(\times)</th>
<th>4</th>
<th>5i</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>12</td>
<td>15i</td>
</tr>
<tr>
<td>2i</td>
<td>8i</td>
<td>10i^2 = -10</td>
</tr>
</tbody>
</table>

Completing the square is a way to solve quadratic equations by forcing them into the form in which you can solve via the method of square roots.

To solve \(x^2 + 6x + 2 = 0\), first notice this:

<table>
<thead>
<tr>
<th>(\times)</th>
<th>(x)</th>
<th>+</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(x^2)</td>
<td>(3x)</td>
<td></td>
</tr>
<tr>
<td>+</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(3x)</td>
<td>(3^2)</td>
<td></td>
</tr>
</tbody>
</table>
This gives us

\[(x + 3)^2 = x^2 + 3x + 3x + 3^2\]
\[= x^2 + 6x + 3^2\]

Looking at it another way, \((x + 3)^2\) is almost equal to \(x^2 + 6x\) except for that pesky \(3^2\).

2nd trick: To solve \(x^2 + 6x + 2 = 0\), (i.e. to find the zeros of the function \(x^2 + 6x + 2\), add that pesky \(3^2\) to both sides just so we can make one side into a square:

\[
x^2 + 6x + 2 = 0 \\
x^2 + 6x + 3^2 + 2 = 3^2 \\
x^2 + 6x + 3^2 = 3^2 - 2 \\
(x + 3)^2 = 3^2 - 2 \\
x + 3 = \pm \sqrt{3^2 - 2} \\
x = -3 \pm \sqrt{3^2 - 2} \\
= -3 \pm \sqrt{7}
\]

The thing on the right is messy, but it’s just a pair of numbers (no \(x\)).

Check the plus version (minus is similar):

\[
x^2 + 6x + 2 = \left(-3 + \sqrt{7}\right)^2 + 6 \left(-3 + \sqrt{7}\right) + 2 \\
= 9 - 3\sqrt{7} - 3\sqrt{7} + 7 - 18 + 6\sqrt{7} + 2 \\
= 9 + 7 - 18 + 2 - 6\sqrt{7} + 6\sqrt{7} \\
= 0
\]

This can be messy but it always works!

If \(a \neq 1\), just divide both sides by \(a\) first.

You can get a quadratic function into vertex form by completing the square with the function definition:

\[
y = x^2 - 8x + 11 \\
y + \left(-\frac{8}{2}\right)^2 = x^2 - 8x + \left(-\frac{8}{2}\right)^2 + 11 \\
y + 16 = x^2 - 8x + 16 + 11 \\
y + 16 = (x - 4)^2 + 11 \\
y = (x - 4)^2 - 5
\]

The vertex is \((4, 5)\).

The minimum of a quadratic whose \(a\) term is positive, or the maximum of a quadratic whose \(a\) term is negative, the vertex. So to find the max or min, just find the vertex.

The **quadratic formula** is just the result of completing the square on the generic quadratic equation, \(ax^2 + bx + c = 0\). This gives us solutions \((x\) values that make that equation true\): \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\). The red part, the radicand or \(b^2 - 4ac\), is called the **discriminant**. If the discriminant is positive, the equation has two real solutions; if the discriminant is zero, there is one solution; if the discriminant is negative there are no real solutions (but two imaginary ones).