Quadratic test study guide

Use this as a reminder of the things you should know for the test. It is not a substitute for practice!

The test covers sections 5.1-5.6 in the book.

The forms (e.g. standard form, vertex form, intercept form etc.) are very precise. For example, the vertex form is $y = a(x - h)^2 + k$. If you have something like $2\,(x + 3)^2 - 7$ that is not exactly in the form! You should force the operation after the x to be subtraction and the operation before k to be addition, converting $y = 2\,(x + 3)^2 - 7$ to $y = 2\,(x - (-3))^2 + (-7)$. Now it's exactly in vertex form so you can say that the vertex is $(-3, -7)$. You may do this in your head, but unless you are sure of yourself, write it down to make errors less likely. For another example, the quadratic expression $x^2 - 2\,x + 3$ can be rewritten as $1\,x^2 + (-2)\,x + 3$ to exactly fit standard form (see below).

Quadratic expression: an *expression* that can be written in the form $a\,x^2 + b\,x + c$.

Quadratic function: a *function* that can be written in the form $y = a\,x^2 + b\,x + c$ (or $f(x) = a\,x^2 + b\,x + c$). This form is also called the **standard form**.

Quadratic equation: an *equation* that can be written in the form $0 = a\,x^2 + b\,x + c$.

Quadratic formula: The formula that gives the solution(s) of a quadratic equation in standard form: $x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$

Note the similarities and differences between an quadratic *expressions*, *equations*, *functions* and the quadratic *formula*.

The graph of a quadratic function in standard form is a parabola:

- Opens up if $a > 0$, down if $a < 0$
- Wider than $y = x^2$ if $|a| < 1$ and narrower if $|a| > 1$.
- x-coordinate of vertex is $-\frac{b}{2\,a}$ (note similarity with quadratic formula)
- Axis of symmetry is the vertical line $x = -\frac{b}{2\,a}$

Other forms of quadratic functions:

- **Vertex form**: $y = a(x - h)^2 + k$. The vertex is (h, k). Axis of symmetry is $x = h$.
- **Intercept form**: $y = a\,(x - p)(x - q)$. The x-intercepts (also called zeroes) are p and q. Axis of symmetry is halfway between p and q: $x = \frac{p + q}{2}$
- Both forms: opens up if $a > 0$, down if $a < 0$

Factoring quadratic expressions:

- To factor $x^2 + b\,x + c$, find (if you can) m and n where $b = m + n$ and $c = m\,n$. Then $(x + m)\,(x + n) = x^2 + b\,x + c$. Example: To factor $x^2 + 5\,x + 6$, note that $5 = 2 + 3$ and $6 = 2 \cdot 3$. Thus $x^2 + 5\,x + 6 = (x + 2)\,(x + 3)$
- To factor $a\,x^2 + b\,x + c$ it's harder; keep trying until you get something that multiplies out correctly. Example: $3\,x^2 - 17\,x + 10 = (3\,x - 2)\,(x - 5)$

- Special patterns:

    - **Difference of two squares**: $a^2 x^2 - b^2 = (a\,x + b)\,(a\,x - b)$. Example: $x^2 - 9 = (x+3)\,(x-3)$

    - **Perfect square trinomial**: $x^2 + 2\,p\,x + p^2 = (x+p)^2$. Example: $x^2 + 6\,x + 9 = (x+3)^2$

Zero product property: If $a\,b = 0$ then $a = 0$ or $b = 0$ (or both). This enables us to solve quadratic equations by factoring.

Solving an equation: finding values of variable(s) that make it true.

We can use the zero product property to **solve a quadratic equation by factoring**:

Example: Solve $x^2 + 3\,x - 18 = 0$. Solution: Factor as $(x+6)\,(x-3) = 0$. Zero product property tells us that $x + 6 = 0$ or $x - 3 = 0$. Thus the solutions (values of x that make it true) are -6 and 3.

Finding the zeros of a quadratic function is just solving the equation that sets the function value to zero. For example, finding the zeros of the function $y = x^2 + 3\,x - 18$ is just solving $x^2 + 3\,x - 18 = 0$ (see above).

A number r is a **square root** of a number s if $r^2 = s$. Every positive real number has two square roots. The positive square root of s is written $\sqrt{s}$; the negative square root is written $-\sqrt{s}$. The symbol $\sqrt{\ }$ is called the **radical sign** and the number under it is the **radicand**. An expression with a radical sign and radicand underneath, like $\sqrt{s}$, is a **radical**.

If $a \geq 0$ and $b \geq 0$, then we have the **Product property**: $\sqrt{a\,b} = \sqrt{a}\,\sqrt{b}$ and **Quotient property**: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Note that there are *no* similar sum and difference properites: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ and $\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$. For example, $\sqrt{1+4} \neq \sqrt{1} + \sqrt{4}$ (work it out).

Solving quadratic equations using square roots: Try to get a square on both sides of the equation and then take square roots, being careful about positive and negative. Usually one side is an obvious square and the other side is a number, which is also an obvious, if ugly, square. Example:

$$\begin{aligned} 2\,x^2 + 1 &= 17 \\ 2\,x^2 &= 16 \\ x^2 &= 8 \\ x &= \pm\sqrt{8} \\ x &= \pm 2\sqrt{2} \end{aligned}$$

Complex numbers

i

Define $i = \sqrt{-1}$, so $i^2 = -1$.

Properties:

- $\sqrt{-r} = i\,\sqrt{r}$

- For example, $i\,\sqrt{3} = \sqrt{-3}$

Example: solve quadratic equation

Solve $3x^2 + 10 = -26$.

$$\begin{aligned} 3x^2 + 10 &= -26 \\ 3x^2 &= -36 \\ x^2 &= -12 \\ x &= \pm\sqrt{-12} \\ x &= \pm i\sqrt{12} \\ x &= \pm i\sqrt{4}\sqrt{3} \\ x &= \pm 2i\sqrt{3} \end{aligned}$$

Standard form of complex number

$$a + b\,i$$

Where a, b are real numbers. a is the *real part* and $b\,i$ is the *imaginary part*.

Arithmetic

Adding/subtracting: Just think of i as a variable, like x:

$$\begin{aligned} (4 + 5i) + (3 + 2i) &= (4 + 3 + 5i + 2i) \\ &= (7 + 7i) \end{aligned}$$

Multiplying: Think of i as a variable, except that it can do tricks.

$$\begin{aligned} (4 + 5i)(3 + 2i) &= 4 \cdot 3 + 4 \cdot 2i + 3 \cdot 5i + 5 \cdot 2i^2 \\ &= 12 + 8i + 15i + 10i^2 \\ &= 12 + 23i + 10i^2 \\ &= 12 + 23i + (-10) \\ &= 2 + 23i \end{aligned}$$

Here is the same multiplication using the distributive law rectangle:

$\times$	4	$5i$
3	12	$15i$
$2i$	$8i$	$10i^2 = -10$

Completing the square is a way to solve quadratic equations by forcing them into the form in which you can solve via the method of square roots.

To solve $x^2 + 6x + 2 = 0$, first notice this:

$\cdot$	x	$+$	3
x	x^2		$3x$
$+$			
3	$3x$		3^2

This gives us

$$\begin{aligned}(x+3)^2 &= x^2+3\,x+3\,x+3^2\\ &= x^2+6\,x+3^2\end{aligned}$$

Looking at it another way, $(x+3)^2$ is almost equal to $x^2+6\,x$ except for that pesky 3^2.

2nd trick: To solve $x^2+6\,x+2=0$, (i.e. to find the zeros of the function $x^2+6\,x+2$, add that pesky 3^2 to both sides just so we can make one side into a square:

$$\begin{aligned}x^2+6x+2 &= 0\\ x^2+6\,x+3^2+2 &= 3^2\\ x^2+6\,x+3^2 &= 3^2-2\\ (x+3)^2 &= 3^2-2\\ x+3 &= \pm\sqrt{3^2-2}\\ x &= -3\pm\sqrt{3^2-2}\\ &= -3\pm\sqrt{7}\end{aligned}$$

The thing on the right is messy, but it's just a pair of numbers (no x).

Check the plus version (minus is similar):

$$\begin{aligned}x^2+6x+2 &= \left(-3+\sqrt{7}\right)^2+6\left(-3+\sqrt{7}\right)+2\\ &= 9-3\sqrt{7}-3\sqrt{7}+7-18+6\,\sqrt{7}+2\\ &= 9+7-18+2-6\,\sqrt{7}+6\,\sqrt{7}\\ &= 0\end{aligned}$$

This can be messy but it always works!

If $a\neq 1$, just divide both sides by a first.

You can get a quadratic function into vertex form by completing the square with the function definition:

$$\begin{aligned}y &= x^2-8\,x+11\\ y+\left(-\frac{8}{2}\right)^2 &= x^2-8\,x+\left(-\frac{8}{2}\right)^2+11\\ y+16 &= x^2-8\,x+16+11\\ y+16 &= (x-4)^2+11\\ y &= (x-4)^2-5\end{aligned}$$

The vertex is $(4,5)$.

The minimum of a quadratic whose a term is positive, or the maximum of a quadratic whose a term is negative, the vertex. So to find the max or min, just find the vertex.

The **quadratic formula** is just the result of completing the square on the generic quadratic equation, $a\,x^2+b\,x+c=0$. This gives us *solutions* (x values that make that equation true): $x=\frac{-b\pm\sqrt{b^2-4\,a\,c}}{2\,a}$. The red part, the radicand or $b^2-4\,a\,c$, is called the **discriminant**. If the discriminant is positive, the equation has two real solutions; if the discriminant is zero, there is one solution; if the discriminant is negative there are no real solutions (but two imaginary ones).