# Completing the square

Warmup: Solve: $x^2+6x+2=0$.

Hard or impossible to factor.

Look at $x^2+6\,x$. (Ignore the 2 for the moment.)

Notice this:

| $\cdot$ | $x$ | $+$ | 3 |
|---|---|---|---|
| $x$ | $x^2$ | | $3\,x$ |
| $+$ | | | |
| 3 | $3\,x$ | | $3^2$ |

This gives us

$$\begin{aligned}(x+3)^2 &= x^2+3\,x+3\,x+3^2\\ &= x^2+6\,x+3^2\end{aligned}$$

Looking at it another way, $(x+3)^2$ is almost equal to $x^2+6\,x$ except for that pesky $3^2$.

2nd trick: To solve $x^2+6\,x+2=0$, (i.e. to find the zeros of the function $x^2+6\,x+2$:

$$\begin{aligned}x^2+6x+2 &= 0\\ x^2+6\,x+3^2+2 &= 3^2\\ x^2+6\,x+3^2 &= 3^2-2\\ (x+3)^2 &= 3^2-2\\ x+3 &= \pm\sqrt{3^2-2}\\ x &= -3\pm\sqrt{3^2-2}\\ &= -3\pm\sqrt{7}\end{aligned}$$

The thing on the right is messy, but it's just a number (no $x$).

Check the plus version (minus is similar):

$$\begin{aligned}x^2+6x+2 &= \left(-3+\sqrt{7}\right)^2+6\left(-3+\sqrt{7}\right)+2\\ &= 9-3\sqrt{7}-3\sqrt{7}+7-18+6\sqrt{7}+2\\ &= 9+7-18+2-6\sqrt{7}+6\sqrt{7}\\ &= 0\end{aligned}$$

This can be messy but it always works!

If $a\neq 1$, just divide both sides by $a$ first:

$$\begin{aligned}3\,x^2-6\,x+12 &= 0\\ x^2-2\,x+4 &= 0\\ x^2-2\,x &= -4\\ x^2-2\,x+(-1)^2 &= -4+(-1)^2\\ (x-1)^2 &= -3\\ x-1 &= \pm\sqrt{-3}\\ x &= 1\pm\sqrt{-3}\\ x &= 1\pm i\sqrt{3}\end{aligned}$$

You can get a quadratic function into vertex form by completing the square with the function definition:

$$\begin{aligned} y &= x^2 - 8x + 11 \\ y + \left(-\frac{8}{2}\right)^2 &= x^2 - 8x + \left(-\frac{8}{2}\right)^2 + 11 \\ y + 16 &= x^2 - 8x + 16 + 11 \\ y + 16 &= (x-4)^2 + 11 \\ y &= (x-4)^2 - 5 \end{aligned}$$

The vertex is $(4, 5)$.

The minimum of a quadratic whose $a$ term is positive, or the maximum of a quadratic whose $a$ term is negative, the vertex. So to find the max or min, just find the vertex.