Complex numbers

i

Define i as $\sqrt{-1}$, so $i^2 = -1$.

Properties:

- $\sqrt{-r} = i\sqrt{r}$
- For example, $i\sqrt{3} = \sqrt{-3}$
    - Why?

$$\begin{aligned} (i\sqrt{3})^2 &= i^2(\sqrt{3})^2 \\ &= i^2 3 \\ &= -1 \cdot 3 \\ &= -3 \\ &= (\sqrt{-3})^2 \end{aligned}$$

Example: solve quadratic equation

Solve $3x^2 + 10 = -26$.

$$\begin{aligned} 3x^2 + 10 &= -26 \\ 3x^2 &= -36 \\ x^2 &= -12 \\ x &= \pm\sqrt{-12} \\ x &= \pm i\sqrt{12} \\ x &= \pm i\sqrt{4}\sqrt{3} \\ x &= \pm 2i\sqrt{3} \end{aligned}$$

Standard form

$$a + b\,i$$

Where a, b are real numbers. a is the *real part* and $b\,i$ is the *imaginary part*.

Plotting complex numbers

Arithmetic

Adding/subtracting: Just think of i as a variable, like x:

$$\begin{aligned} (4 + 5i) + (3 + 2i) &= (4 + 3 + 5i + 2i) \\ &= (7 + 7i) \end{aligned}$$

Multiplying: Think of i as a variable, except that it can do tricks.

$$\begin{aligned}(4+5\,i)(3+2\,i) &= 4\cdot 3+4\cdot 2i+3\cdot 5\,i+5\cdot 2\,i^2\\ &= 12+8\,i+15\,i+10i^2\\ &= 12+23\,i+10\,i^2\\ &= 12+23\,i+(-10)\\ &= 2+23\,i\end{aligned}$$

Here is the same multiplication using the distributive law rectangle:

$\times$	4	$5\,i$
3	12	$15\,i$
$2\,i$	$8\,i$	$10\,i^2=-10$

Complex conjugate of $a+b\,i$ is $a-b\,i$.

Multiplying by the conjugate is guaranteed to produce a real number:

$$\begin{aligned}(a+b\,i)(a-b\,i) &= a^2-a\,b\,i+a\,b\,i-b^2i^2\\ &= a^2-b^2\,(-1)\\ &= a^2+b^2\end{aligned}$$

Example: dividing complex numbers.

Write $\frac{5+3\,i}{1-2\,i}$ in standard form. Solution:

$$\begin{aligned}\frac{5+3\,i}{1-2\,i} &= \frac{5+3\,i}{1-2\,i}\cdot\frac{1+2\,i}{1+2\,i}\\ &= \frac{5+13\,i-6}{1+4}\\ &= \frac{13\,i-1}{5}\\ &= -\frac{1}{5}+\frac{13\,i}{5}\end{aligned}$$

Absolute value:

$$|a+b\,i|=\sqrt{a^2+b^2}$$

Example:

$$\begin{aligned}|3+4\,i| &= \sqrt{3^2+4^2}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

Note that the absolute value can also be written in terms of the complex conjugate:

$$|a+b\,i|=\sqrt{(a+b\,i)\,(a-b\,i)}$$

Here's the same example, using the conjugate form:

$$\begin{aligned}|3+4\,i| &= \sqrt{(3+4\,i)\,(3-4\,i)}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

Geometrically, the absolute value of a real number is the distance of that real number from zero on the number line. Similarly, $|a + b\,i|$ is the distance of $a + b\,i$ from 0 in the complex plane, using the Pythagorean theorem to find the hypotenuse of a right triangle with legs of length a and b.