Chapter 4 Review: Matrices

We did a quick tour of chapter 4. We covered sections 4.1-4.4. Here are what you should be able to do:

1. Know that an $m \times n$ matrix has m rows and n columns. Rows are horizontal and columns are vertical.

2. Add matrices. This is done by adding *corresponding entries*. For example:

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & -6 & 4 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 5 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2+\frac{1}{2} & -1+5 & 3+0 \\ 0+0 & -6+1 & 4+2 \end{pmatrix} = \begin{pmatrix} 2\frac{1}{2} & 4 & 3 \\ 0 & -5 & 6 \end{pmatrix}$$

   Addition is only possible when the the matrices have the same dimensions, i.e. they have the same number of rows and columns.

3. Subtract matrices. This works the same as addition except that you subtract corresponding entries:

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 3-(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

4. Multiply a matrix by a *scalar* (a number not in a matrix). This is done by multiplying the scalar by each entry:

$$\frac{1}{3}\begin{pmatrix} 6 & -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\cdot 6 & \frac{1}{3}\cdot(-3) & \frac{1}{3}\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & \frac{1}{3} \end{pmatrix}$$

5. Compare matrices: Matrices are equal if all corresponding entries are equal. So $\begin{pmatrix} 2+3 & 3 \end{pmatrix} = \begin{pmatrix} 5 & \frac{6}{2} \end{pmatrix}$ but $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$.

6. Multiply matrices. The product of two matrices is a matrix in which a given entry p in the product consists of the sum of the products of all the entries in the same row as p in the first matrix and all the entries in the same column as p in the second matrix. So for example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+2\cdot 4 & 1\cdot 2+2\cdot 5 & 1\cdot 3+2\cdot 6 \\ 3\cdot 1+4\cdot 4 & 3\cdot 2+4\cdot 5 & 3\cdot 3+4\cdot 6 \\ 5\cdot 1+6\cdot 4 & 5\cdot 2+6\cdot 5 & 5\cdot 3+6\cdot 6 \end{pmatrix} = \begin{pmatrix} 1 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}$$

   It may help to write the first matrix to the left of the product and the second matrix above the product. That way an arrow through a row in the first matrix and an arrow through a column in the second matrix together point to the entry that is the sum of the products of the entries in that row and that column:

$$\begin{array}{cc} \times & \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix} \end{array}$$

7. Know that the product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix, and that the number of columns in the first matrix must equal the number of rows in the second matrix. You can figure this out by just knowing how to multiply.

8. Know that the determinant of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is written $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and its value is $a\,d - b\,c$. (The determinant of a matrix is a number, not a matrix.). So for example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

9. Know that the 2×2 *identity matrix* is written I; that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; and that $I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

10. Know how to use the multiplicative inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ of a 2×2 matrix; that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I$. You will not need to remember that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

but you will need to be able to use that fact. So for example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\frac{1}{2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

What kinds of matrices have no multiplicative inverse?

11. Know how to use Cramer's rule:

$$\text{If} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{then} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}\begin{pmatrix} e\,d - f\,b \\ a\,f - c\,e \end{pmatrix}$$

to solve 2×2 systems of quations, either written as matrix equations. So for example the solution to

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

is the same as the solution to

$$\begin{aligned} x + 2\,y &= 5 \\ 3\,x + 4\,y &= 6 \end{aligned}$$

and that solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}}\begin{pmatrix} 5 \cdot 4 - 6 \cdot 2 \\ 1 \cdot 6 - 3 \cdot 5 \end{pmatrix} = -\frac{1}{2} \cdot \begin{pmatrix} 8 \\ -9 \end{pmatrix} = \begin{pmatrix} -4 \\ \frac{9}{2} \end{pmatrix}$$

This solution can also be written as $x = -4$; $y = \frac{9}{2}$.