Cramer's Rule

Today we'll use matrices to derive Cramer's Rule, a rule by which we can solve systems of equations by just plugging in numbers. (Cramer's Rule works for $n \times n$ matrices but we'll use the 2×2 case only.)

Let's start with a generic system of two linear equations in two unknowns:

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \tag{1}$$

Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $X = \begin{pmatrix} x \\ y \end{pmatrix}$; $E = \begin{pmatrix} e \\ f \end{pmatrix}$.

We can see that the equation $A\,X = E$ is equivalent to the system (1):

$$\begin{aligned} A\,X &= E \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} e \\ f \end{pmatrix} \\ \begin{pmatrix} a\,x + b\,y \\ c\,x + d\,y \end{pmatrix} &= \begin{pmatrix} e \\ f \end{pmatrix} \end{aligned}$$

Finding a solution means finding values for x and y that make it true, or in the language of matrices, finding a value for X that makes it true. Well, we already have a single (matrix) equation involving X:

$$A\,X = E$$

How do we solve it? If we were solving a simple equation about numbers, we could divide both sides by A, or equivalently, multiply both sides by $\frac{1}{A}$. With matrices, we can do almost the same thing: multiply both sides by A^{-1}:

$$A^{-1}\,A\,X = A^{-1}\,E \tag{2}$$

And, just like, say, $\frac{1}{3} \cdot 3 = 1$, it is also true that $A^{-1}\,A = I$ so $A^{-1}\,A\,X = I\,X = X$. Thus we can simplify equation (2) thus:

$$X = A^{-1}E \tag{3}$$

This tells us our solution! Because we learned recently that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|\,A\,|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we can work out the matrix multiplication:

$$\begin{aligned} X &= A^{-1}E \\ &= \frac{1}{|\,A\,|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} \\ &= \frac{1}{|\,A\,|}\begin{pmatrix} d\,e - b\,f \\ a\,f - c\,e \end{pmatrix} \end{aligned} \tag{4}$$

This (4) is Cramer's Rule.

Example

Solve the system

$$\begin{aligned} 8\,x + 5\,y &= 2 \\ 2\,x - 4\,y &= -10 \end{aligned}$$

Solution (using Cramer's Rule):

$$\begin{aligned} X &= \frac{1}{\left|\begin{smallmatrix} 8 & 5 \\ 2 & -4 \end{smallmatrix}\right|}\begin{pmatrix} 2(-4)-5(-10) \\ 8(-10)-2(2) \end{pmatrix} \\ &= \frac{1}{8(-4)-2\cdot 5}\begin{pmatrix} 42 \\ -84 \end{pmatrix} \\ &= \frac{1}{-42}\begin{pmatrix} 42 \\ -84 \end{pmatrix} \\ &= \begin{pmatrix} \frac{42}{-42} \\ \frac{-84}{-42} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}$$

Now we should check that we have a solution. We can either plug into the system or into the matrix form. Let's try the latter.

$$\begin{aligned} \begin{pmatrix} 8 & 5 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} &= \begin{pmatrix} 8(-1)+5\cdot 2 \\ 2(-1)+(-4)2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -10 \end{pmatrix} \end{aligned}$$

Whew. Let's try one more: Solve the system

$$\begin{aligned} 2x+y &= 3 \\ 5x+6y &= 4 \end{aligned}$$

So let $A=\left(\begin{smallmatrix} 2 & 1 \\ 5 & 6 \end{smallmatrix}\right)$; $X=\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$; $E=\left(\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right)$, and we wish to solve $AX=E$. Cramer's Rule says

$$\begin{aligned} X &= \frac{1}{2\cdot 6-5\cdot 1}\begin{pmatrix} 6\cdot 3-1\cdot 4 \\ 2\cdot 4-5\cdot 3 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 14 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

Let's check: $\left(\begin{smallmatrix} 2 & 1 \\ 5 & 6 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)=\left(\begin{smallmatrix} 2\cdot 2+1(-1) \\ 5\cdot 2+6(-1) \end{smallmatrix}\right)=\left(\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right)$. Whew again.