Introduction to Logarithms

1. The inverse of the exponential function 2^x, is called a *logarithm* function. In particular it is called the *logarithm base 2*. We write it as $\log_2$. So for example we may write $\log_2(8) = 3$, which is another way of saying $2^3 = 8$. Often we omit the function parentheses and write it $\log_2 8 = 3$. Below are some exponential equations and their equivalent logarithm versions. (The symbol $\Leftrightarrow$ means "is equivalent to.") Fill in the missing equations. The first one is already done for you.

$$\begin{array}{lcl} 2^3 = 8 & \Leftrightarrow & \log_2 8 = 3 \\ 2^4 = 16 & \Leftrightarrow & \log_2 16 = 4 \\ 2^{-3} = \frac{1}{8} & \Leftrightarrow & \log_2 \frac{1}{8} = -3 \\ 2^0 = 1 & \Leftrightarrow & \log_2 1 = 0 \\ 2^1 = 2 & \Leftrightarrow & \log_2 2 = 1 \end{array}$$

2. The above logarithms have a *base* of 2. A logarithm function asks this question of its input: What power of the base yields the logarithm's input value? For example, to find $\log_2 16$ we ask, what power of 2 yields 16? One way to solve it is to realize that question is equivalent to solving $2^x = 16$. So what is $\log_2 16$?

3. There are also logarithms with other bases. The *common logarithm* has a base of 10. Sometimes we write common logarithms without an explicit base, so for example $\log 1000 = \log_{10} 1000$. What is the value of $\log 1000$? In other words, what power of 10 yields 1000?

4. We can summarize the above material with the following two equivalences:

$$\begin{array}{rcl} \log_b x = y & \Leftrightarrow & b^y = x \\ \log x = y & \Leftrightarrow & 10^y = x \end{array}$$

Change of base

1. The change-of-base formula is this:

$$\log_b x = \frac{\log_c x}{\log_c b}$$

In our course we will usually use the case where $c = 10$:

$$\log_b x = \frac{\log x}{\log b}$$

What this allows us to do is to compute logs in any base if we can compute logs in any other base. Our calculators compute logs in base 10, not in most other bases. So we can use the change-of-base formula to compute logs in other bases.

- Example: Let's say we want to compute $\log_3 81$ with our calculators (of course we should be able to do this one in our heads). To use the calculator, note that

$$\log_3 81 = \frac{\log 81}{\log 3}$$

To get this with the calculator, just do `log(81)/log(3)`. That should give the answer you expect. (Use the log key for the base 10 logarithms.)

- For the curious, here's a way to derive the change of base formula. fFirst notice this equality:

$$\begin{aligned} c^{\log_c b \log_b x} &= (c^{\log_c b})^{\log_b x} \\ &= b^{\log_b x} \\ &= x \\ &= c^{\log_c x} \end{aligned}$$

Summarizing the above, we see that

$$c^{\log_c b \log_b x} = c^{\log_c x}$$

Since each side of the equation is a power of c, we see that the powers must also be equal:

$$\log_c b \log_b x = \log_c x$$

Dividing both sides by $\log_c b$ we get the change-of-base formula:

$$\log_b x = \frac{\log_c x}{\log_c b}$$

Other logarithm properties

$$\begin{aligned} \log_b(uv) &= \log_b u + \log_b v \\ \log_b(\frac{u}{v}) &= \log_b u - \log_b v \\ \log_b(u^n) &= n \log_b u \end{aligned}$$

For the curious, we can derive these properties by using each side of the equation as a power of b and using exponent properties. For example, we can derive the log of the product this way:

$$\begin{aligned} b^{\log_b u + \log_b v} &= b^{\log_b u} b^{\log_b v} \\ &= uv \\ &= b^{\log_b(uv)} \end{aligned}$$

The log of the quotient can be derived similarly. (Try it!) Here's the log of a power:

$$\begin{aligned} b^{n \log_b u} &= (b^{\log_b u})^n \\ &= u^n \\ &= b^{\log_b(u^n)} \end{aligned}$$

Condensing and expanding logarithms

Condensing logarithms involves using the logarithm properties to convert a log expression to one containing a single logarithm. For example:

$$\begin{aligned} 3 \log x - \log y &= \log x^3 - \log y \\ &= \log(\frac{x^3}{y}) \end{aligned}$$

Expanding goes the other way:

$$\begin{aligned}\log(\frac{x^3}{y}) &= \log x^3 - \log y \\ &= 3\log x - \log y3\end{aligned}$$

"Holy Grail" of a logarithm equation

If you can get an equation containing logarithms on both sides of an equation to consist of logarithms of the same base, i.e. of the form

$$\log_b x = \log_b y$$

then you can conclude that $x = y$. For example:

$$\begin{aligned}2\log_7 x &= \log_7(6-x) \\ \log_7(x^2) &= \log_7(6-x) \text{ (Holy Grail!)} \\ x^2 &= 6-x \\ x^2+x-6 &= 0 \\ (x+3)(x-2) &= 0 \\ x &= -3, 2\end{aligned}$$

Estimating logarithms

Often we can estimate logarithms if the input number is not far from a power of the base. For example, to estimate $\log_3 79$, we notice that $3^4 = 81$ which is just a bit higher than 79. So $\log_3 79 \approx 3.99$ (a little less than 4).

Using logarithms to solve exponential equations

Since logs ask the "what power" question, they are good for solving exponential equations:

$$\begin{aligned}5^x &= 1000 \\ \log_5(5^x) &= \log_5 1000 \\ x &= \log_5 1000 \\ x &\approx 4.29 \text{ (using change-of-base on calculator)}\end{aligned}$$

Check: $5^{4.29} \approx 996.7 \approx 1000$