Introduction to Logarithms

1. Make a table for the function $f(x) = 2^x$. Use at least $-4, -3, -2, -1, 0, 1, 2, 3, 4$ for x values.

2. Graph that function on graph paper. Make sure that x and y are on the same scale. Both the x and y scale will have to include at least -4 through 2^4.

3. Now make another table representing the function $f^{-1}(x)$ by making a table just like the other one but with the two columns switched (so for example $f^{-1}(8) = 3$). Remember that the notation f^{-1} here represents the *inverse function* of f, which has nothing to do with raising the function value to the -1 power. This is just an unfortunate ambiguity in the language of mathematics.

4. Graph f^{-1}. If all is correct, the graph of f^{-1} looks just like the graph of f reflected about the line $y = x$, like any inverse function.

5. The function f^{-1}, the inverse of the exponential function 2^x, is called a *logarithm* function. In particular it is called the *logarithm base 2*. We write it as $\log_2$. So for example we may write $\log_2(8) = 3$, which is another way of saying $2^3 = 8$. Often we omit the function parentheses and write it $\log_2 8 = 3$. Below are some exponential equations and their equivalent logarithm versions. (The symbol $\Leftrightarrow$ means "is equivalent to.") Fill in the missing equations. The first one is already done for you.

$$2^3 = 8 \quad \Leftrightarrow \quad \log_2 8 = 3$$
$$2^4 = 16 \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \log_2 \tfrac{1}{8} = -3$$
$$2^0 = 1 \quad \Leftrightarrow$$
$$\Leftrightarrow \quad \log_2 2 = 1$$

6. The above logarithms have a *base* of 2. A logarithm function asks this question of its input: What power of the base yields the logarithm's input value? For example, to find $\log_2 16$ we ask, what power of 2 yields 16? One way to solve it is to realize that question is equivalent to solving $2^x = 16$. So what is $\log_2 16$?

7. There are also logarithms with other bases. The *common logarithm* has a base of 10. Sometimes we write common logarithms without an explicit base, so for example $\log 1000 = \log_{10} 1000$. What is the value of $\log 1000$? In other words, what power of 10 yields 1000?

8. We can summarize the above material with the following two equivalences:

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x$$
$$\log x = y \quad \Leftrightarrow \quad 10^y = x$$

Change of base

1. The change-of-base formula is this:

$$\log_b x = \frac{\log_c x}{\log_c b}$$

In our course we will usually use the case where $c = 10$:

$$\log_b x = \frac{\log x}{\log b}$$

What this allows us to do is to compute logs in any base if we can compute logs in any other base. Our calculators compute logs in base 10, not in most other bases. So we can use the change-of-base formula to compute logs in other bases.

- Example: Let's say we want to compute $\log_3 81$ with our calculators (of course we should be able to do this one in our heads). To use the calculator, note that

  $$\log_3 81 = \frac{\log 81}{\log 3}$$

  To get this with the calculator, just do `log(81)/log(3)`. That should give the answer you expect. (Use the $\boxed{\log}$ key for the base 10 logarithms.)

- For the curious, here's a way to derive the change of base formula. First notice this equality:

  $$\begin{aligned} c^{\log_c b \log_b x} &= (c^{\log_c b})^{\log_b x} \\ &= b^{\log_b x} \\ &= x \\ &= c^{\log_c x} \end{aligned}$$

  Summarizing the above, we see that

  $$c^{\log_c b \log_b x} = c^{\log_c x}$$

  Since each side of the equation is a power of c, we see that the powers must also be equal:

  $$\log_c b \log_b x = \log_c x$$

  Dividing both sides by $\log_c b$ we get the change-of-base formula:

  $$\log_b x = \frac{\log_c x}{\log_c b}$$

Other logarithm properties

$$\begin{aligned} \log_b(uv) &= \log_b u + \log_b v \\ \log_b(\frac{u}{v}) &= \log_b u - \log_b v \\ \log_b(u^n) &= n \log_b u \end{aligned}$$

For the curious, we can derive these properties by using each side of the equation as a power of b and using exponent properties. For example, we can derive the log of the product this way:

$$\begin{aligned} b^{\log_b u + \log_b v} &= b^{\log_b u} b^{\log_b v} \\ &= uv \\ &= b^{\log_b (uv)} \end{aligned}$$

The log of the quotient can be derived similarly. (Try it!) Here's the log of a power:

$$\begin{aligned} b^{n \log_b u} &= (b^{\log_b u})^n \\ &= u^n \\ &= b^{\log_b (u^n)} \end{aligned}$$