Polynomial Graphing Quiz Study Guide, 2009-02-03

This is a quick study guide for the quiz this Friday. This will only make sense to you if you have done the work!

- **End behavior**: What happens to the graph as $x \to \infty$ and as $x \to -\infty$
  - Right end behavior: look at positive x first: if the coefficient of highest-degree term is *positive*, $f(x) \to \infty$ (goes *up*) as $x \to \infty$. If the coefficient is *negative*, $f(x) \to -\infty$ (goes *down*) as $x \to \infty$.
  - Left end behavior: If *highest degree* is *even*, then $f(x)$ goes in the *same direction* on the left end as on the right end (think parabola). But if the highest degree is *odd*, then $f(x)$ goes in the *opposite direction* on the left end as on the right end (think cubic or even linear).
  - Example: $-2x^3 + 7x^2$ has a negative coefficient (-2) on the highest-degree term, so it goes down as $x \to \infty$ (right end). Because the degree, 3, is odd, it goes in the opposite direction when $x \to -\infty$ (left end) so in this case it goes up on the left. For another example, $5x^4 - 3x$ has a positive coefficient (5) and an even degree, so $5x^4$ goes up at both ends.
- **Finding zeros**: Factor the polynomial as much as possible (unless already factored). Zeros corresponding to linear factors like $x - 3$ are easy (in that case the zero is 3). If you have a quadratic you can't factor, use other quadratic methods (like square roots or quadratic formula) to find zeros. For example, to find the zeros of $x^3 + x^2 - x$, first factor as $x(x^2 + x - 1)$. The zero corresponding to x is just 0, and the zeros corresponding to $x^2 + x - 1$ are $\frac{-1 \pm \sqrt{5}}{2}$ based on the quadratic formula, so altogether the zeros are 0, $\frac{-1 + \sqrt{5}}{2}$ and $\frac{-1 - \sqrt{5}}{2}$.
- **Multiplicity** of a zero: This is the number of times its corresponding factor appears in the factorization. In $(x+2)^3(x-4)^2(x+5)$, there are three zeros: -2, 4, and -5. The multiplicity of the zero -2 is 3 because in the factorization, $x + 2$ appears three times (written as $(x+2)^3$ but could also be written $(x+2)(x+2)(x+2)$. Similarly, the multiplicity of the zero 4 is 2 and the multiplicity of the zero -5 is 1.
- **How to sketch a polynomial graph**: Start by finding the degree and leading coefficient. From there, determine the end behavior. Sketch the left and right ends of the graph. Factor the polynomial as much as possible, if necessary. Determine the zeros based on the factors (e.g. if $(x-3)$ is a factor, 3 is a zero.) Determine the multiplicity of each zero. Mark the zeros on the x-axis.Then, starting from the end behavior piece on the left side, approach the first zero. If it's a zero of multiplicity 1 (linear term), cross through the zero in something close to a straight line. If it has *even muliplicity*, do not cross, but rather *bounce* at the zero back in the direction you came from, making a little almost-parabola (technically it's not parabolic if it's a quartic or some other even degree greater than two, but it's close enough for our purposes). If it has odd multiplicity, 3 or greater, cross the x-axis with a little "flex" or "twist", the way a cubic looks. After you finish the rightmost zero, connect it to the right end behavior. You get a free error check there, because after the rightmost zero, you should be on the same side of the x-axis as the right end behavior piece.
- **How to infer a polynomial from its graph**: First, look at the end behavior to see if it has even or odd degree and whether the coefficient is positive or negative (see "end behavior" above). Then find the zeros. For each zero h, create a factor $(x-h)$. Multiply those together. For every zero that is a bounce, raise that factor to some even power. For every zero that gets crossed with a flex, raise that factor to an odd power greater than or equal to 3. For every zero that gets crossed approximately linearly, leave that factor at the first power. Now look at the end behavior and determine whether the whole polynomial is of even or odd degree and whether the leading coefficient is posi-

tive or negative. Hopefully the problem will have guidelines so you don't have to try too many possibilities (e.g. all bounces are multiplicity of 2, or the degree of the polynomial is less than whatever).

- If you can manage to be sure that all your factors are right, and if you get at least one y-intercept or other point, you can figure it out precisely: just plug in the information you know into your equation. For example, if you are told it's of degree 6, if it has end behavior going up at both ends, a linear zero at -5, a flex at -2, and a bounce at 4, what do we know? The linear zero at -5 means a factor of $(x+5)$; the flex at -2 means a factor of $(x+2)$ and an odd multiplicity of at least 3; the bounce at 4 means a factor of $(x-4)$ and even multiplicity. Since we are told the degree is 6, that means the multiplicity of the zero -2 must be exactly 3 and the multiplicity of the zero 4 must be exactly 2. All together this means we have factors $(x+5)(x+2)^3(x-4)^2$. But all we know about the coefficient is that it is positive. But if we are given a point, say a y-intercept of 1280, we can plug in that information:

$$\begin{aligned} a(0+5)(0+2)^3(0-4)^2 &= 1280 \\ a\cdot 5\cdot 8\cdot 16 &= 1280 \\ a\cdot 640 &= 1280 \\ a &= 2 \end{aligned}$$

  This tells us the whole function is $f(x)=2(x+5)(x+2)^3(x-4)^2$.